

Analytic treatment of leading-order parton evolution equations: theory and tests

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We recently derived an explicit expression for the gluon distribution function $G(x, Q^2) = xg(x, Q^2)$ in terms of the proton structure function $F_2^{\gamma p}(x, Q^2)$ in leading-order (LO) QCD by solving the LO DGLAP equation for the Q^2 evolution of $F_2^{\gamma p}(x, Q^2)$ analytically, using a differential-equation method. We showed that accurate experimental knowledge of $F_2^{\gamma p}(x, Q^2)$ in a region of Bjorken x and virtuality Q^2 is all that is needed to determine the gluon distribution in that region. We re-derive and extend the results here using a Laplace-transform technique, and show that the singlet quark structure function $F_S(x, Q^2)$ can be determined directly in terms of G from the DGLAP gluon evolution equation. To illustrate the method and check the consistency of existing LO quark and gluon distributions, we used the published values of the LO quark distributions from the CTEQ5L and MRST2001LO analyses to form $F_2^{\gamma p}(x, Q^2)$, and then solved analytically for $G(x, Q^2)$. We find that the analytic and fitted gluon distributions from MRST2001LO agree well with each other for all x and Q^2 , while those from CTEQ5L differ significantly from each other for large x values, $x \gtrsim 0.03-0.05$, at all Q^2 . We conclude that the published CTEQ5L distributions are incompatible in this region. Using a non-singlet evolution equation, we obtain a sensitive test of quark distributions which holds in both LO and NLO perturbative QCD. We find in either case that the CTEQ5 quark distributions satisfy the tests numerically for small x , but fail the tests for $x \gtrsim 0.03-0.05$ —their use could potentially lead to significant shifts in predictions of quantities sensitive to large x . We encountered no problems with the MRST2001LO distributions or later CTEQ distributions. We suggest caution in the use of the CTEQ5 distributions.

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I. INTRODUCTION

In a recent paper [1], we derived an explicit expression for the gluon distribution function $G(x, Q^2) = xg(x, Q^2)$ in the proton in terms of the proton structure function $F_2^{\gamma p}(x, Q^2)$ for γ^*p scattering. The result was obtained in leading-order (LO) QCD by solving the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation [2] for the Q^2 evolution of $F_2^{\gamma p}(x, Q^2)$ analytically, assuming massless quarks and using a differential-equation method. The method is model-independent and does not require individual quark distributions, so accurate experimental knowledge of $F_2^{\gamma p}(x, Q^2)$ in a region of Bjorken x and virtuality Q^2 is all that is needed to determine the gluon distribution in that region.

In the present paper, we extend the method using a Laplace-transform technique. We first re-derive the solution for $G(x, Q^2)$ in terms of

$$F_2^{\gamma p}(x, Q^2) = \sum_{i=1}^{n_f} e_i^2 x [q_i(x, Q^2) + \bar{q}_i(x, Q^2)] \quad (1)$$

using our Laplace transform method. The same procedure can be used to derive $G(x, Q^2)$ from the singlet (S) quark distribution function

$$F_S(x, Q^2) = \sum_{i=1}^{n_f} x [q_i(x, Q^2) + \bar{q}_i(x, Q^2)] \quad (2)$$

and other combinations of the quark distributions. We then obtain an analytic solution for $F_S(x, Q^2)$ in terms

of $G(x, Q^2)$ by solving the LO gluon evolution equation exactly. These are the principal theoretical results in this paper.

To illustrate the method, we have used the LO CTEQ5L [3] and MRST2001 [4] quark distributions to construct the proton γ^*p and singlet structure functions as functions of x and Q^2 , solved analytically for the corresponding gluon distributions, and compared the results for $G(x, Q^2)$ with those given in the fits. We find that the analytic and published distributions agree to high accuracy for MRST2001 LO, but, to our surprise, that they are incompatible for CTEQ5L, differing significantly at large x for all Q^2 . We find the same situation when we use the evolution equations for the singlet quark distribution or for individual quarks to solve for $G(x, Q^2)$. We have not encountered these difficulties in somewhat less-extensive calculations using the more recent CTEQ6L [5] distributions, finding good agreement between the analytic and published gluon distributions.

To investigate the inconsistencies further, we have used the DGLAP evolution equation for the non-singlet (NS) distributions

$$F_{NS} \equiv \sum_{i=1}^{n_f} x [u_i(x, Q^2) + \bar{u}_i(x, Q^2) - d_i(x, Q^2) - \bar{d}_i(x, Q^2)], \quad (3)$$

where the sum is over the up- and down-type quarks and antiquarks in each generation with both types active. As will be shown, F_{NS} can be used in both leading order

(LO) and next-to-leading order (NLO) to construct sensitive tests of the quark distributions.

The tests based on F_{NS} are satisfied for the MRST2001 LO [4] and CTEQ6L [5] quark distributions which also satisfied the gluon tests above to high accuracy. However, we find that the LO CTEQ5L and the NLO CTEQ5M quark distributions in the $\overline{\text{MS}}$ renormalization scheme fail to satisfy the NS evolution equation in the same regions at large x where the gluons gave difficulty. This shows clearly that these published quark distributions are inconsistent at large x .

The errors in the quark distributions and their incompatibilities with G could well affect predictions for experimental quantities that are sensitive to large x . We conclude that the CTEQ5L distributions, though very convenient because of the analytic parametrizations given [3], should not be used to make precise predictions at large x .

II. PRELIMINARIES

We first introduce a quantity $\mathcal{F}_2(x, Q^2)$, which depends *only* on the proton structure function $F_2^{\gamma p}(x, Q^2)$, by

$$\begin{aligned} \mathcal{F}_2(x, Q^2) \equiv & \frac{\partial F_2^{\gamma p}(x, Q^2)}{\partial \ln(Q^2)} \\ & - \frac{\alpha_s}{4\pi} \left\{ 4F_2^{\gamma p}(x, Q^2) + \frac{16}{3} \left[F_2^{\gamma p}(x, Q^2) \ln \frac{1-x}{x} \right. \right. \\ & + x \int_x^1 \left(\frac{F_2^{\gamma p}(z, Q^2)}{z} - \frac{F_2^{\gamma p}(x, Q^2)}{x} \right) \frac{dz}{z-x} \Big] \\ & \left. - \frac{8}{3} x \int_x^1 F_2^{\gamma p}(z, Q^2) \left(1 + \frac{x}{z} \right) \frac{dz}{z^2} \right\}. \end{aligned} \quad (4)$$

We define the corresponding quantities $\mathcal{F}_S(x, Q^2)$ and $\mathcal{F}_{NS}(x, Q^2)$ for the singlet and non-singlet quark distributions obtained by replacing $F_2^{\gamma p}$ by F_S or F_{NS} .

The LO DGLAP evolution equations [2] for the proton structure functions $F_2^{\gamma p}(x, Q^2)$, $F_S(x, Q^2)$, and $F_{NS}(x, Q^2)$ for massless quarks can be written compactly in x space as

$$\mathcal{F}_2(x, Q^2) = \frac{\alpha_s}{4\pi} \sum_i e_i^2 x \int_x^1 G(z, Q^2) K_{qg} \left(\frac{x}{z} \right) \frac{dz}{z^2}, \quad (5)$$

$$\mathcal{F}_S(x, Q^2) = 2n_f \frac{\alpha_s}{4\pi} x \int_x^1 G(z, Q^2) K_{qg} \left(\frac{x}{z} \right) \frac{dz}{z^2}, \quad (6)$$

$$\mathcal{F}_{NS}(x, Q^2) \equiv 0, \quad (7)$$

where the sum over quark charges in Eq. (5) includes both quarks and antiquarks, and n_f in Eq. (6) is the number of active quarks. \mathcal{F}_{NS} vanishes only if both members of a quark family are massless or can otherwise be treated as active. The LO $g \rightarrow q$ splitting function is given by

$$K_{qg}(x) = 1 - 2x + 2x^2. \quad (8)$$

Since

$$F_2^{\gamma p}(x, Q^2) = \frac{5}{18} F_S(x, Q^2) + \frac{1}{6} F_{NS}(x, Q^2), \quad (9)$$

we should be able to derive $G(x, Q^2)$ consistently in LO from either of Eqs. (5) or (6) provided Eq. (7) is satisfied.

We note that Eqs. (6) and (7) continue to hold at next-to-leading order with modified quark and gluon splitting functions in Eqs. (4) and (6). These are given in the $\overline{\text{MS}}$ scheme in Eqs. (4.7a), (4.7b) and (4.7c) of Floratos *et al.* [6].

To simplify the equations, we rewrite Eq. (5) as

$$\mathcal{F}\mathcal{F}(x, Q^2) = x \int_x^1 G(z, Q^2) K_{qg} \left(\frac{x}{z} \right) \frac{dz}{z^2}, \quad (10)$$

where we have introduced $\mathcal{F}\mathcal{F}(x, Q^2)$, which is defined as

$$\mathcal{F}\mathcal{F}(x, Q^2) = \left(\frac{\alpha_s}{4\pi} \sum_i e_i^2 \right)^{-1} \mathcal{F}_2(x, Q^2). \quad (11)$$

Similarly, we define $\mathcal{F}\mathcal{F}_S(x, Q^2)$ in terms of the singlet quark distribution $F_S(x, Q^2)$ by

$$\mathcal{F}\mathcal{F}_S(x, Q^2) = \left(2n_f \frac{\alpha_s}{4\pi} \right)^{-1} \mathcal{F}_S(x, Q^2), \quad (12)$$

and analogous quantities with appropriate prefactors for other structure functions such as $F_{2(3)}^{\gamma Z}$, $F_{2(3)}^Z$, and $F_{2(3)}^{W^\mp}$.

We note also that if we wish to evolve a single massless quark distribution in LO, we can simply replace $F_2^{\gamma p}$ in Eq. (4) by $Q_i = xq_i(x, Q^2)$ and $\mathcal{F}\mathcal{F}$ in Eq. (11) by the appropriate

$$\mathcal{F}\mathcal{F}_i(x, Q^2) = \left(\frac{\alpha_s}{4\pi} \right)^{-1} \mathcal{F}_i(x, Q^2), \quad (13)$$

so that

$$\mathcal{F}\mathcal{F}_i(x, Q^2) = x \int_x^1 G(z, Q^2) K_{qg} \left(\frac{x}{z} \right) \frac{dz}{z^2}, \quad (14)$$

with $i = u, \bar{u}, d, \bar{d}, c, \bar{c}, s, \bar{s}$.

Since the gluon content of Eq. (14) is independent of the type of quark, Eq. (14) has the important consequence that, in LO,

$$\frac{\mathcal{F}\mathcal{F}_i(x, Q^2)}{\mathcal{F}\mathcal{F}_j(x, Q^2)} = 1, \quad (15)$$

for all x at a fixed virtuality Q^2 , and any choice of i and j . As we will soon see, this turns out to be a very sensitive test of compatibility of quark distributions with the LO QCD evolution equations.

An alternate form of this relation, which is readily extendable to NLO, follows from the evolution equation for the non-singlet distribution $F_{NS}(x, Q^2)$, $\mathcal{F}\mathcal{F}_{NS}(x, Q^2) = 0$. Splitting the sums in Eq. (3) into sums over up-

and down-type quarks, we find that the vanishing of $\mathcal{FF}_{\text{NS}}(x, Q^2)$ implies that

$$\frac{\mathcal{FF}_{\text{dtype}}(x, Q^2)}{\mathcal{FF}_{\text{utype}}(x, Q^2)} = 1 \quad (16)$$

for *all* x at any given virtuality Q^2 . This furnishes us with an exact numerical test of NLO perturbative QCD. We will later see that, while the MRST2001 and CTEQ6 parton distributions satisfy our constraints for all Q^2 and x , the LO CTEQ5L and NLO $\overline{\text{MS}}$ CTEQ5M quark distributions satisfy these tests for small x , but fail badly at large x .

III. SOLUTION OF THE LO EVOLUTION EQUATION FOR THE PROTON STRUCTURE FUNCTION $F_2^{\gamma p}(x, Q^2)$

We now re-derive our analytic solution [1] of Eq. (10) for the LO gluon distribution, $G(x, Q^2)$ using a new Laplace-transform method. Introducing the coordinate transformation

$$v \equiv \ln(1/x), \quad (17)$$

we define functions \hat{G} , \hat{K}_{qg} , and $\hat{\mathcal{F}}$ in v -space by

$$\begin{aligned} \hat{G}(v, Q^2) &\equiv G(e^{-v}, Q^2) \\ \hat{K}_{qg}(v) &\equiv K_{qg}(e^{-v}) \\ \hat{\mathcal{F}}(v, Q^2) &\equiv \mathcal{FF}(e^{-v}, Q^2). \end{aligned} \quad (18)$$

Explicitly, from Eq. (8), we see that

$$\hat{K}_{qg}(v) = 1 - 2e^{-v} + 2e^{-2v}. \quad (19)$$

Further, we can write

$$\begin{aligned} \hat{\mathcal{F}}(v, Q^2) &= \int_0^v \hat{G}(w, Q^2) e^{-(v-w)} \hat{K}_{qg}(v-w) dw \\ &= \int_0^v \hat{G}(w, Q^2) \hat{H}(v-w) dw, \end{aligned} \quad (20)$$

where

$$\begin{aligned} \hat{H}(v) &\equiv e^{-v} \hat{K}_{qg}(v) \\ &= e^{-v} - 2e^{-2v} + 2e^{-3v}. \end{aligned} \quad (21)$$

We introduce the notation that the Laplace transform of a function $\hat{H}(v)$ is given by $h(s)$, where

$$h(s) \equiv \mathcal{L}[\hat{H}(v); s] = \int_0^\infty \hat{H}(v) e^{-sv} dv, \quad (22)$$

with the condition $\hat{H}(v) = 0$ for $v < 0$. The convolution theorem for Laplace transforms relates the transform of a convolution of functions \hat{G} and \hat{H} to the product of their transforms $g(s)$ and $h(s)$, so that

$$\mathcal{L}\left[\int_0^v \hat{G}(w) \hat{H}(v-w) dw; s\right] = g(s) \times h(s), \quad (23)$$

or conversely, the inverse transform of a product to the convolution of the original functions, giving

$$\begin{aligned} \mathcal{L}^{-1}[g(s) \times h(s); v] &= \int_0^v \hat{G}(w) \hat{H}(v-w) dw \\ &= \int_0^v \hat{H}(w) \hat{G}(v-w) dw. \end{aligned} \quad (24)$$

Equation (23) allows us to write the Laplace transform of Eq. (10) as

$$f(s, Q^2) = g(s, Q^2) \times h(s), \quad (25)$$

where

$$f(s, Q^2) = \mathcal{L}[\hat{\mathcal{F}}(v, Q^2); s] \quad (26)$$

and $g(s, Q^2)$ and $h(s)$ are the transforms of the functions \hat{G} and \hat{H} on the right-hand-side of Eq. (20). Solving Eq. (25) for g , we find that

$$g(s, Q^2) = f(s, Q^2)/h(s). \quad (27)$$

We will generally not be able to calculate the inverse transform of $g(s, Q^2)$ explicitly, if only because $f(s, Q^2)$ is determined by a numerical integral of the experimentally-determined function $\hat{\mathcal{F}}(v, Q^2)$. However, if we regard $g(s, Q^2)$ as the product of the two functions $f(s, Q^2)$ and $h^{-1}(s)$ and take the inverse Laplace transform using the convolution theorem in the form in Eq. (24) and the *known inverse* $\mathcal{L}^{-1}[f(s, Q^2); v] = \hat{\mathcal{F}}(v, Q^2)$, we find that

$$\begin{aligned} \hat{G}(v, Q^2) &= \mathcal{L}^{-1}[f(s, Q^2) \times h^{-1}(s); v] \\ &= \int_0^v \hat{\mathcal{F}}(w, Q^2) \hat{J}(v-w) dw \end{aligned} \quad (28)$$

where $\hat{J}(v)$ is a new auxiliary function, defined by

$$\hat{J}(v) \equiv \mathcal{L}^{-1}[h^{-1}(s); v]. \quad (29)$$

The calculation of $h(s)$ and the inverse Laplace transform of $h^{-1}(s)$ are straightforward, and we find that

$$\begin{aligned} \hat{J}(v) &= 3\delta(v) + \delta'(v) \\ &\quad - e^{-3v/2} \left(\frac{6}{\sqrt{7}} \sin \left[\frac{\sqrt{7}}{2} v \right] \right. \\ &\quad \left. + 2 \cos \left[\frac{\sqrt{7}}{2} v \right] \right). \end{aligned} \quad (30)$$

We therefore obtain an explicit solution for the gluon distribution $G(v, Q^2)$ in terms of the integral

$$\begin{aligned} \hat{G}(v, Q^2) &= \int_0^v \hat{\mathcal{F}}(w, Q^2) \hat{J}(v-w) dw \\ &= 3\hat{\mathcal{F}}(v, Q^2) + \frac{\partial \hat{\mathcal{F}}(v, Q^2)}{\partial v} \\ &\quad - \int_0^v \hat{\mathcal{F}}(w, Q^2) e^{-3(v-w)/2} \times \end{aligned}$$

$$\left(\frac{6}{\sqrt{7}} \sin \left[\frac{\sqrt{7}}{2} (v-w) \right] + 2 \cos \left[\frac{\sqrt{7}}{2} (v-w) \right] \right) dw. \quad (31)$$

Transforming Eq. (31) back into x space, we find that the gluon probability distribution $G(x, Q^2)$ is given in terms of the function $\mathcal{F}\mathcal{F}$, assumed to be known, by

$$G(x, Q^2) = 3\mathcal{F}\mathcal{F}(x, Q^2) - x \frac{\partial \mathcal{F}\mathcal{F}(x, Q^2)}{\partial x} - \int_x^1 \mathcal{F}\mathcal{F}(z, Q^2) \left(\frac{x}{z} \right)^{3/2} \times \left\{ \frac{6}{\sqrt{7}} \sin \left[\frac{\sqrt{7}}{2} \ln \frac{z}{x} \right] + 2 \cos \left[\frac{\sqrt{7}}{2} \ln \frac{z}{x} \right] \right\} \frac{dz}{z}. \quad (32)$$

It can be shown without difficulty that this expression for G is equivalent to that which we derived in [1], where we first converted the evolution equation for $F_2^{\gamma p}(x, Q^2)$ into a differential equation in v , and then solved that equation explicitly.

Even if we cannot integrate the last term of Eq. (32) analytically, the usual case, it can be integrated numerically, so that we *always* have an explicit solution which can be evaluated to the numerical accuracy to which $\mathcal{F}\mathcal{F}(x, Q^2)$ is known. We emphasize that this solution for G is derived from $F_2^{\gamma p}$ on the assumption that the quarks are either massless, or effectively so, a situation that holds in general for $Q^2 \gg 4M_i^2$, where mass effects due to the i^{th} (massive) quark are negligible, or for a treatment of mass effects such as that used in CTEQ5 [3] as discussed in Sec. V.¹

Leading-order solutions for $G(x, Q^2)$ of exactly the same form follow from the evolution equations for the singlet structure function F_S and the individual quark distribution functions, with $\mathcal{F}\mathcal{F}(x, Q^2)$ replaced in Eq. (32) by $\mathcal{F}\mathcal{F}_S(x, Q^2)$ or $\mathcal{F}\mathcal{F}_i(x, Q^2)$, as noted earlier. The latter can be used to obtain expressions for $G(x, Q^2)$ in terms of the structure functions for weak scattering processes, allowing the incorporation of other data sets.

IV. SOLUTION OF THE LO GLUON EVOLUTION EQUATION

To obtain an expression which relates the singlet structure function $F_S(x, Q^2)$ indirectly to experiment, we use the DGLAP equation for the Q^2 evolution of the gluon distribution $G(x, Q^2)$. This relates F_S to G which is

determined by $F_2^{\gamma p}$ as shown in the preceding section. Again, we simplify the notation by writing this equation in terms of the (rescaled) $q \rightarrow g$ splitting function $K_{gq}(x)$ and a quantity $\mathcal{G}\mathcal{G}(x, Q^2)$ as

$$\mathcal{G}\mathcal{G}(x, Q^2) = x \int_x^1 F_S(x, Q^2) K_{gq}(x/z) \frac{dz}{z^2}. \quad (33)$$

Here

$$K_{gq}(x) \equiv \frac{2}{x} - 2 + x, \quad (34)$$

and

$$\mathcal{G}\mathcal{G}(x, Q^2) \equiv \frac{3}{8} \left\{ \left(\frac{\alpha_s}{4\pi} \right)^{-1} \frac{\partial G(x, Q^2)}{\partial \ln Q^2} - G(x, Q^2) \left(\frac{1}{3} (33 - 2n_f) + 12 \ln(1-x) \right) - 12x \left[\int_x^1 G(z, Q^2) \left(\frac{z}{x} - 2 + \frac{x}{z} - \left(\frac{x}{z} \right)^2 \right) \frac{dz}{z^2} + \int_x^1 (G(z, Q^2) - G(x, Q^2)) \left(\frac{z}{z-x} \right) \frac{dz}{z^2} \right] \right\}, \quad (35)$$

for n_f effectively massless active quark flavors. As before, going to v space, we define the quantities

$$\begin{aligned} \hat{F}_S(v, Q^2) &\equiv F_S(e^{-v}, Q^2), \\ \hat{\mathcal{G}}(v, Q^2) &\equiv \mathcal{G}\mathcal{G}(e^{-v}, Q^2), \\ \hat{H}_{gq}(v, Q^2) &\equiv e^{-v} K_{gq}(e^{-v}) \\ &= e^{-v} (2e^v - 2 + e^{-v}) \\ &= 2 - 2e^{-v} + e^{-2v}. \end{aligned} \quad (36)$$

The evolution equation then involves a convolution of \hat{F}_S and \hat{H}_{gq} .

Proceeding as before, we take the Laplace transform of both sides, solve for the transform f_S of \hat{F}_S , invert the transform using the convolution theorem again, and find that

$$\hat{F}_S(v, Q^2) = \frac{\partial \hat{\mathcal{G}}(v, Q^2)}{\partial v} + \int_0^v \hat{\mathcal{G}}(w, Q^2) e^{-3(v-w)/2} \times \left\{ \frac{6}{\sqrt{7}} \sin \left[\frac{\sqrt{7}}{2} (v-w) \right] - 2 \cos \left[\frac{\sqrt{7}}{2} (v-w) \right] \right\} dw. \quad (37)$$

Transforming back into x space, $F_S(x, Q^2)$ is given by

$$F_S(x, Q^2) = -x \frac{\partial \mathcal{G}\mathcal{G}(x, Q^2)}{\partial x} + \int_x^1 \mathcal{G}\mathcal{G}(z, Q^2) \left(\frac{x}{z} \right)^{3/2} \times \left\{ \frac{6}{\sqrt{7}} \sin \left[\frac{\sqrt{7}}{2} \ln \frac{z}{x} \right] - 2 \cos \left[\frac{\sqrt{7}}{2} \ln \frac{z}{x} \right] \right\} \frac{dz}{z}. \quad (38)$$

¹ We will discuss mass effects in detail elsewhere.

$\mathcal{GG}(x, Q^2)$ is a known function of the gluon distribution, $G(x, Q^2)$, which, in turn, is a known function of the proton structure function $F_2^{\gamma p}(x, Q^2)$ that is measured in deep inelastic scattering. $F_S(x, Q^2)$ is therefore related back to $F_2^{\gamma p}(x, Q^2)$ through $G(x, Q^2)$ and the analysis in the preceding section, and could be used, in turn, to determine $G(x, Q^2)$ through Eq. (32) with \mathcal{FF}_S replacing \mathcal{FF} .

It is important to recognize that our method is essentially model-independent. When different parametrizations in x and Q^2 are used to fit experimental $F_2^{\gamma p}$ data, the LO gluon and F_S distributions derived from the different fits must agree with each other to the accuracy of the fits in the regions in which the data exist. The results can only depend on the functional forms used to fit $F_2^{\gamma p}(x, Q^2)$ when they are extrapolated to x and Q^2 outside that region.

V. NUMERICAL ILLUSTRATION OF THE METHOD, AND TESTS OF LO GLUON AND QUARK DISTRIBUTIONS

To test of our methods, we have applied them to published parton distributions, using the quark distributions to construct $F_2^{\gamma p}$ and F_S and then solving Eq.(32) to find G . The methods work well, and reproduce the published gluons distributions except in the case of CTEQ5 [3]. In that case, we found to our surprise that there are problems at high x with the LO CTEQ5L distributions. While these are superseded by more recent CTEQ distributions [5, 7], they seem still to be used in some calculations, most likely because they have been parametrized analytically in a form convenient for calculation. Similar tests applied to the MRST2001 [4] and the CTEQ6L [5] LO distributions uncovered no problems.

The results illustrate the sensitivity of the analytic methods in testing parton distributions, and may be of wider use for that purpose, as well as for deriving G directly from data as originally proposed in [1]. We present them here as a demonstration, and as a caution.

We begin by illustrating the use of the analytic expression in Eq. (32) to derive $G(x, Q^2)$ from $F_2^{\gamma p}(x, Q^2)$ in the case of CTEQ5L [3]. We take the published LO CTEQ5L quark distributions as our basic input, and use these distributions to calculate the proton structure function $F_2^{\gamma p}(x, Q^2) = \sum_i e_i^2 x [q_i(x, Q^2) + \bar{q}_i(x, Q^2)]$ needed in $\mathcal{FF}(x, Q^2)$ in Eq. (11). Next, we solve Eq. (32) for the G generated by this $F_2^{\gamma p}$, labelling the solution $G_{\text{Analytic}}(x, Q^2)$, and compare the results with the published gluon distributions.

We follow the procedures of the CTEQ5 group in these calculations. The parton splitting functions used are those for massless quarks. Quark mass effects are included only in an approximate, x -independent way, with xq_i and $x\bar{q}_i$ taken as zero for a massive quark i when $Q^2 < M_i^2$, and the quark treated as fully active when $Q^2 > M_i^2$, with q_i and \bar{q}_i included in the calculations

of the structure functions $F_2^{\gamma p}$ and F_S . The charge factors $\sum_i e_i^2$ in Eqs. (5) and (11), and the numbers of active quarks n_f in Eqs. (6), (12), and later in Eq. (50), then change discontinuously at each threshold, but remain constant between thresholds. Despite these discontinuities, the quark and gluon distributions are continuous.²

Because the mass effects do not depend on x in this approach, the evolution equations retain the massless form between thresholds, and we can simply use the results derived above for massless quarks in our analysis, treating the distinct inter-threshold regions in Q^2 separately.³

In these calculations, we use the LO form of $\alpha_s(Q^2)$ used in CTEQ5L [8],

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0 \ln(Q^2/\Lambda^2)}, \quad (39)$$

$$\beta_0 = 11 - \frac{2}{3}n_f, \quad (40)$$

with $n_f = 5$ and $\Lambda_5 = 146$ MeV for $Q > 4.5$ GeV, $n_f = 4$ and $\Lambda_4 = 192$ MeV for $1.3 \text{ GeV} < Q \leq 4.5$ GeV, and $n_f = 3$ and $\Lambda_3 = 221$ MeV for $Q < 1.3$ GeV. These parameters give a continuous $\alpha_s(Q^2)$ with $\alpha_s(M_Z^2) = 0.127$ [3].

The same methods can be used in LO to determine the gluon distributions G_i generated by individual quark distributions, or by other combinations of the quarks, in particular, by combinations of the massless quarks u , d , s and by the non-singlet distributions, as will be seen below.

Following these procedures, the gluon distribution $G_{\text{Analytic}}(x, Q^2)$ we derive analytically from the calculated structure functions $F_2^{\gamma p}$ or F_S , or from other quark combinations, should agree to the accuracy of the calculations with $G_{\text{CTEQ5L}}(x, Q^2)$ at all x and Q^2 .

The results of the calculations based on $F_2^{\gamma p}$ are shown in Fig. 1. The left-hand column compares G_{Analytic} (blue dashed curves) with G_{CTEQ5L} (red solid curves) at $Q^2 = 5, 20$, and 100 GeV^2 , (a), (b), and (c), respectively. All

² The quark and gluon distributions are continuous solutions of the evolution equations, which are first-order differential equations in $\ln Q^2$ and integral equations in x . The discontinuities on the right-hand sides of Eqs. (5) and (6) as n_f changes at a threshold are reflected in the final results by discontinuous changes in the derivatives of the structure functions $\partial F / \partial \ln Q^2$ on the left-hand sides of these equations. In particular, the heavy quark distributions $xq_i(x, Q^2)$ and $x\bar{q}_i(x, Q^2)$ are identically zero for $Q^2 < M_i^2$, then initially rise linearly from zero with increasing $\ln Q^2$ for $Q^2 \geq M_i^2$. The contributions of $\partial(xq_i)_i / \partial \ln Q^2$ and $\partial(x\bar{q}_i)_i / \partial \ln Q^2$ to $\partial F / \partial \ln Q^2$, identically zero for $Q^2 < M_i^2$, therefore jump to non-zero values for $Q^2 \geq M_i^2$.

³ Mass effects are treated the same way by CTEQ6 [5]. The more accurate treatments of heavy-quark masses in MRST2001 [4] and the recent CTEQ analyses beginning with CTEQ6.5 [7] involve x -dependent effects. These will not affect the tests of the MRST2001 distributions given later using only the massless quarks.

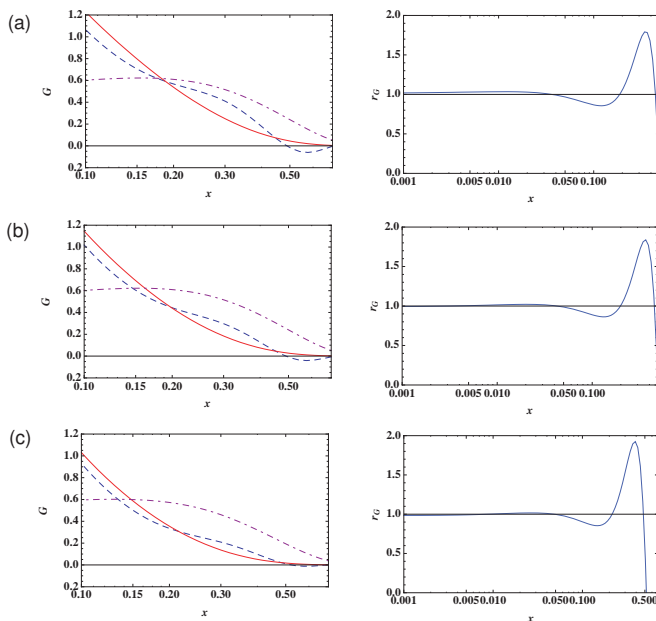


FIG. 1: Left -hand column: comparison of the G_{Analytic} (blue dashed curves) obtained by solving Eq. (32) using the CTEQ5L $F_2^{\gamma p}(x, Q^2)$ as input, with G_{CTEQ5L} (red curves). The u -quark distributions $U(x, Q^2) = xu(x, Q^2)$ (purple dot-dashed curves) are shown to give an independent scale at the same x and Q^2 . (a) $Q^2 = 5 \text{ GeV}^2$, $n_f = 4$. (b) $Q^2 = 20 \text{ GeV}^2$, $n_f = 4$. (c) $Q^2 = 100 \text{ GeV}^2$, $n_f = 5$. Right-hand column: the ratio $r_G = G_{\text{Analytic}}/G_{\text{CTEQ5L}}$ at the same Q^2 as (a), (b), (c). The ratios should equal 1 for all x and Q^2 .

of the active, physically-relevant quarks are included in the input at each value of Q^2 , with $n_f = 4$ for (a) and (b) with u, d, s, c active, and $n_f = 5$ in (c) with b now also active. For a scale comparison, we include the CTEQ5L up-quark distribution, $U(x, Q^2) = xu(x, Q^2)$.

The analytic and fitted G s do not agree well at large x , with significant differences between them on the scale of G and of the u -quark distribution for $0.05 \lesssim x \lesssim 0.6$. We do not believe that the calculations are reliable beyond $x \approx 0.6$: G is very small, but the input u and d distributions and the individual integrals in \mathcal{FF} are not, and the calculation in Eq. (32) involves large cancellations and becomes sensitive to small errors in the inputs.

The same discrepancies between the two G s are illustrated differently in the right-hand column of Fig. 1 where we plot the ratio

$$r_G = G_{\text{Analytic}}(x, Q^2)/G_{\text{CTEQ5L}}(x, Q^2). \quad (41)$$

In all cases, we find that r_G is essentially constant and equal to one for all $x \lesssim 0.01$ for small virtuality, and for all $x \lesssim 0.05$ for large virtuality, showing good agreement between analytic and fitted gluon distributions at small x . There are again significant deviations from one at larger x , contrary to theoretical requirements.

We emphasize that the reason we find *negative* values

of G_{Analytic} or r_G for some $x \gtrsim 0.5$ is *not* that $F_2^{\gamma p}(x, Q^2)$ goes negative, but rather that $\mathcal{FF}(x, Q^2)$ goes negative. As can be seen from Eq. (4), this is a sensitive function of the *difference* between $\partial F_2^{\gamma p}(x, Q^2)/\partial \ln(Q^2)$ and the LO convolution integral of $F_2^{\gamma p}(x, Q^2)$. In spite of the fact that the CTEQ5L quark distributions that go into $F_2^{\gamma p}(x, Q^2)$ are all positive, the resulting combination $\mathcal{FF}(x, Q^2)$ becomes negative for large x . The appearance of negative values for the gluon distribution function suggests strongly that the differences between G_{Analytic} and G_{CTEQ5L} in Fig. 1 result from problems with the input quark distributions, hence with G_{Analytic} , and are less likely to arise directly from problems with G_{CTEQ5L} .

We find very similar deviations if we use the singlet structure function $F_S(x, Q^2)$ in Eq. (2) as input, and then solve for $G(x, Q^2)$ using Eq. (32) with \mathcal{FF}_S replacing \mathcal{FF} .

The discrepancies between the analytic and fitted G s are similar in magnitude to the differences between the CTEQ5 and more recent CTEQ and MRST gluon distributions in this region, and to the changes in the distributions that resulted from the addition of inclusive jet data to those analyses. Changes of this size are clearly significant.

Since the major contributions to $F_2^{\gamma p}(x, Q^2)$ at large x are from the valence quarks u and d , we have done a separate set of calculations using the singlet combination $F_{S,ud} = x(u + \bar{u} + d + \bar{d})$ and Eq. (32) with $\mathcal{FF}_{S,ud}$ replacing \mathcal{FF} . No mass effects need to be included in this case. The results of the calculations, called $G_{ud}(x, Q^2)$, are shown in Fig. 2 for both CTEQ5L and MRST2001 LO.

Discrepancies between G_{CTEQ5L} and the G analytically reconstructed from $F_{S,ud}$, here labeled $G_{ud}(x, Q^2)$, are again evident in the figure. The discrepancies have the same pattern and are roughly the same sizes as those seen in Fig. 1, suggesting that the problems originate with the u and d distributions that drive the initial DGLAP evolution at large x . The results for MRST2001 LO show agreement between the calculated G_{ud} and G_{MRST2001} at the level of accuracy of the calculation, which required the parametrization of numerical data from [9].

To exclude the possibility that the problems with CTEQ5 can be eliminated by going to NLO perturbative QCD, we also looked at the evolution of the non-singlet structure function $F_{NS}(x, Q^2)$, and at the gluon distribution from the evolution of the singlet structure function $F_S(x, Q^2)$, using the CTEQ5M NLO distributions as input. We found a similar x -dependence for the gluon ratio obtained from the NLO singlet distribution, hence, continuing problems. The NS combination, for both LO and NLO, is discussed in the next section, and shows clearly that there are problems with the CTEQ5L quark distributions.

We note finally that, to reduce the possibility that the discrepancies that we have encountered are an artifact of the published parametrization of the CTEQ5L

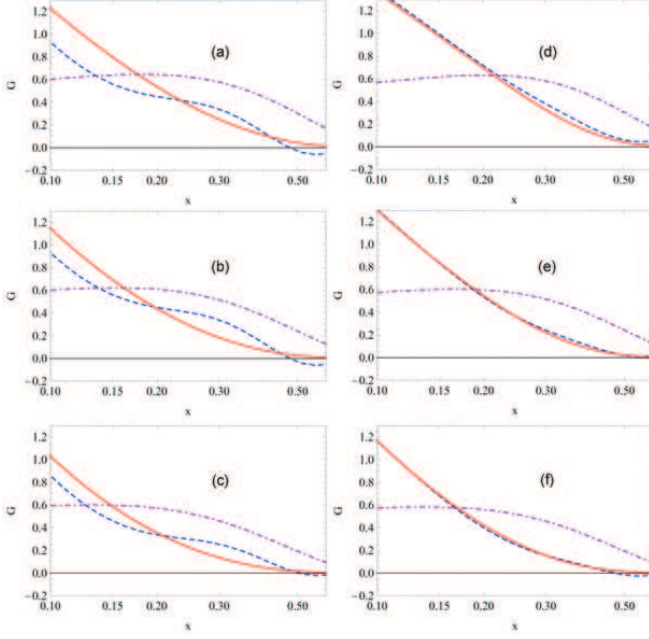


FIG. 2: Plots of the $n_f = 2$ $G_{ud}(x, Q^2)$ (blue dashed curves), $G_{fitted}(x, Q^2)$ (red curves), and the u -quark distribution $U(x, Q^2) = xu(x, Q^2)$ (purple dot dashed curves) vs. x , for $x > 0.1$. Left-hand column: $G_{fitted} = G_{CTEQ5L}$ distributions for (a) $Q^2 = 5$ GeV², (b) $Q^2 = 20$ GeV², (c) $Q^2 = 100$ GeV². Right-hand column: $G_{fitted} = G_{MRST2001}$ LO distributions for (d) $Q^2 = 5$ GeV², (e) $Q^2 = 20$ GeV², (f) $Q^2 = 100$ GeV².

parton distributions [3], we have also compared them to the corresponding numerical CTEQ5L distributions from the Durham web site [9] and concluded that they are numerically compatible.

Because the $G_{Analytic}(x, Q^2)$ derived from either $F_2^{\gamma p}(x, Q^2)$ or $F_S(x, Q^2)$ fails to match $C_{CTEQ5L}(x, Q^2)$ well at large x , we cannot expect the inverse problem, the analytic derivation of F_S from G to work well for the CTEQ5L distributions, and do not include such calculations here.

VI. TESTS OF LO AND NLO NON-SINGLET DISTRIBUTIONS

For massless quarks in LO, the relation in Eq. (15) holds for any quark or anti-quark pair i and j ,

$$\frac{\mathcal{F}\mathcal{F}_i(x, Q^2)}{\mathcal{F}\mathcal{F}_j(x, Q^2)} = 1. \quad (42)$$

As discussed earlier, this can be generalized readily to NLO using the non-singlet evolution equation $\mathcal{F}_{NS}(x, Q^2) = 0$, Eq. (7) with the NLO NS splitting functions of Floratos *et al.* [6]. If we separate the sum in Eq.

(3) into sums over u - and d -type quarks, defined as

$$u_{\text{type}} \equiv x[u(x, Q^2) + \bar{u}(x, Q^2) + c(x, Q^2) + \bar{c}(x, Q^2)], \quad (43)$$

$$d_{\text{type}} \equiv x[d(x, Q^2) + \bar{d}(x, Q^2) + s(x, Q^2) + \bar{s}(x, Q^2)], \quad (44)$$

this relation becomes

$$\mathcal{F}\mathcal{F}_{NS}(x, Q^2) = \mathcal{F}\mathcal{F}_{u\text{type}}(x, Q^2) - \mathcal{F}\mathcal{F}_{d\text{type}}(x, Q^2) = 0, \quad (45)$$

where the two terms are to be calculated separately using the same NLO splitting functions as for the full F_{NS} . Dividing by the u -type term, we obtain the ratio

$$r_{NS} \equiv \frac{\mathcal{F}\mathcal{F}_{d\text{type}}(x, Q^2)}{\mathcal{F}\mathcal{F}_{u\text{type}}(x, Q^2)} = 1, \quad (46)$$

a relation true for both LO and NLO when all the relevant quarks are active.

To use this relation to check the consistency of the quark distributions, we insert the appropriate LO or NLO quark and anti-quark distributions in Eqs. (43) and (44) and use them to generate $r_{NS}(x, Q^2)$ numerically using the LO or NLO splitting functions.

In the case of the NLO CTEQ5M $\overline{\text{MS}}$ distributions, we use

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0 \ln(Q^2/\Lambda^2)} \times \left[1 - \frac{2\beta_1}{\beta_0^2} \frac{\ln[\ln(Q^2/\Lambda^2)]}{\ln(Q^2/\Lambda^2)} \right], \quad (47)$$

$$\beta_0 = 11 - \frac{2}{3}n_f \quad (48)$$

$$\beta_1 = 51 - \frac{19}{3}n_f, \quad (49)$$

with $\alpha_s(M_z^2) = 0.118$ for $n_f = 5$, matching to the α 's for $n_f = 4$ and 3 at $Q = 4.5$ and 1.3 GeV.

In the left-hand column of Fig. 3, we plot r_{NS} vs. x , for $Q^2 = 5, 20$, and 100 GeV² and $n_f = 4$. The solid (red) curves were calculated using the LO CTEQ5L quark distributions as input, and the dashed (blue) curves, the NLO CTEQ5M quark distributions in the $\overline{\text{MS}}$ renormalization scheme. We see that the patterns are very similar for all Q^2 , with ratios $r_{NS} \approx 1$ in absolute normalization for all $x \lesssim 0.01$, and with very large deviations occurring at large x . This result, independent of the value of Q^2 , shows explicitly that there are problems with the quark distributions at large x : the NS evolution equation is not satisfied in either leading or next-to-leading order.

To look at the size of the deviations in a more quantitative way, we introduce the notion of a LO “non-singlet gluon” distribution $G_{NS}(x, Q^2)$, defined by the equation

$$\mathcal{F}_{NS}(x, Q^2) = 2n_f \frac{\alpha_s}{4\pi} x \int_x^1 G_{NS}(z, Q^2) K_{qg}\left(\frac{x}{z}\right) \frac{dz}{z^2}. \quad (50)$$

$G_{NS} = 0$ if $\mathcal{F}_{NS}(x, Q^2)$ is a solution of the non-singlet DGLAP equation for an even number of effectively massless quarks. Further, G_{NS} is determined mainly by the u

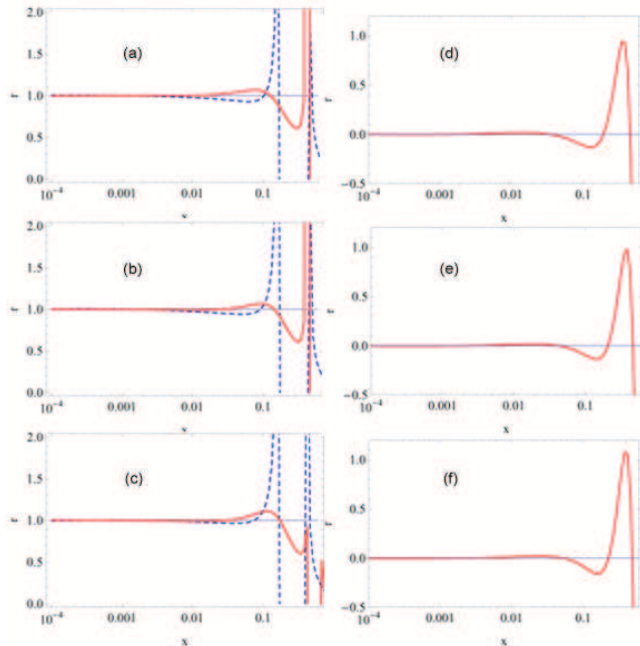


FIG. 3: Left-hand column: the LO and NLO non-singlet d -type to u -type ratio of Eq. (46) for $n_f = 4$ and $Q^2 = 5, 20$, and 100 GeV^2 , (a), (b), and (c), respectively. The ratio should be 1 at all x and Q^2 . The solid (red) curves are LO, calculated from Eq. (43), Eq. (44), and Eq. (16) using CTEQ5L data for quark distributions. The dashed (blue) curves are NLO calculated for the $\overline{\text{MS}}$ CTEQ5M quark distributions [3]. Right-hand column: $r_{NS,G} = G_{NS}(x)/G_{\text{CTEQ5L}}(x)$ vs. x , for $Q^2 = 5, 20$, and 100 GeV^2 , (d), (e), and (f), respectively. The non-singlet “gluon” distribution $G_{NS}(x)$ calculated from u and d quarks is defined in the text in Eq. (50) and is expected to be zero.

and d distributions at large x , so measures the accuracy of those distributions.

The normalization on the right-hand side of Eq. (50) has been chosen so that this equation is the formal counterpart of the singlet equation Eq. (7).

The ratio

$$r_{NS,G} \equiv \frac{G_{NS}(x, Q^2)}{G(x, Q^2)}, \quad (51)$$

should vanish for solutions of the DGLAP equations, and any deviations are scaled to the size of the actual (singlet) gluon distribution G_{CTEQ5L} .

We consider specifically the non-singlet combination of massless u and d quarks,

$$F_{NS,ud}(x, Q^2) \equiv x [u(x, Q^2) + \bar{u}(x, Q^2) - d(x, Q^2) - \bar{d}(x, Q^2)], \quad (52)$$

and calculate $r_{NS,G}$ for that combination, shown in the right-hand column of Fig. 3. Any failure of the ratio to

vanish indicates an inconsistency of the u and d distributions as solutions of the DGLAP evolution equations, and *absolute* magnitudes of the ratio near 1 would indicate that the integrated discrepancies are large.

As seen from the right-hand column in Fig. 3, the ratio $r_{NS,G}$ is zero for small x , but has large absolute values for $x \gtrsim 0.1$ for $Q^2 = 5, 20$, and 100 GeV^2 , Figs. 3 (d), (e), and (f) respectively. In particular, around $x \sim 0.3$, the ratio is about unity, indicating a serious discrepancy near the maxima of the valence-quark distributions.

The pattern in x of the discrepancies between the gluon distributions found analytically from either $F_2^{\gamma p}$ or F_S and the distributions published by the CTEQ5 group closely follows the pattern seen in the departure of \mathcal{FF}_{NS} from zero. For example, \mathcal{FF}_{NS} and the differences between the analytic and fitted gluon distributions all become negative or positive in the same regions. This would seem to indicate that the quark distributions at large x (and, in particular, the dominant u and d valence distributions) are the origin of the difficulty, the combination of their shapes and Q^2 dependence not being compatible with the DGLAP evolution equations, *either in LO or in NLO in the $\overline{\text{MS}}$ renormalization scheme.*

VII. CONCLUSIONS

1. We have first developed a powerful method for the analytic solution of the LO DGLAP evolution equations based on Laplace transforms. This method allows us to determine $G(x, Q^2)$ directly from $F_2^{\gamma p}(x, Q^2)$ or other measurable structure functions such as $F_{2(3)}^{\gamma Z}$, $F_{2(3)}^Z$, and $F_{2(3)}^{W^\mp}$, providing close and simple connections to experiment for massless or effectively massless quarks.

We can also determine $G(x, Q^2)$ analytically from the singlet quark distribution $F_S(x, Q^2)$ if the quark distributions are known, or $F_S(x, Q^2)$ from $G(x, Q^2)$ when the latter is known. The set of relations provide consistency checks on LO quark and gluon distributions obtained in other ways. These are the principal theoretical results of the paper.

2. As an illustration of our methods, we have used the analytic solutions to the evolution equations to obtain tests of the consistency of published quark and gluon distributions. In particular, we compare the gluon distributions determined analytically from structure functions $F_2^{\gamma p}(x, Q^2)$ and $F_S(x, Q^2)$ calculated from the quark distributions determined in various analyses, to the gluon distributions given by the same analyses. We have found no problems for analytic and fitted gluon distributions for the MRST2001 LO or CTEQ6L solutions, while the analytic and fitted gluon distributions for CTEQ5, though consistent at small x , show small, but significant, deviations in the large x region, indicating that the quark and gluons distributions are not

completely consistent [10].

3. A further analysis using the non-singlet structure function $F_{NS}(x, Q^2)$ and both the LO CTEQ5L and the NLO CTEQ5M quark distributions showed definitively that there are problems with those distributions at high x . The quark distributions are not consistent with either LO (CTEQ5L) or NLO perturbative QCD in the $\overline{\text{MS}}$ scheme (CTEQ5M) in the sense that they do not satisfy the appropriate non-singlet relations for $x \gtrsim 0.05$ even though they satisfy them very well at small x . The discrepancies, of unknown origin, are large for $x \sim 0.3$, and could have serious consequences for predictions of processes sensitive to that x region. We conclude that the CTEQ5 distributions [3] should not be used to make such predictions. Again, we found no problems with the MRST2001 distributions [4] or the more recent CTEQ6 distributions [5, 7].

4. We suggest that groups that calculate parton distributions numerically from the coupled DGLAP equations construct the appropriate *analytic solutions* to test the consistency of their results.

We note, finally, that we are working on an extension of our analytic LO solution for $G(x, Q^2)$ to include massive c and b quarks, using the methods of [7], and on the NLO gluon solution.

Acknowledgments

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